

Long Tail of Quantum Decay from Scattering Data

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Abstract. Whereas the short time behaviour of an unstable quantum mechanical system is well understood from its theoretical as well as experimental side, the long time tail of the very same systems has neither been measured experimentally nor is there a theoretical agreement on how to handle it. We suggest a possible way out of this unsatisfactory state of art. Theoretically we suggest that the correct spectral function entering the Fock-Krylov method to calculate the survival amplitude is proportional to the density of states of a resonance. The latter is essentially the energy derivative of a phase shift. As a bonus, we can connect the survival probability to scattering data via the phase shift. The method then not only establishes the spectral function, but is per se a semi-empirical method to extract the large time behaviour from scattering data.

Keywords: non-exponential decay, large time behaviour of quantum systems, broad resonances

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A SHORT HISTORICAL TALE OF QUANTUM DECAY

This year, in 2008, we celebrate the fiftieth anniversary of a famous quantum mechanical result which has spawned numerous papers on the subject. In 1958 Leonid Khalfin proved that in the quantum world, the exponential decay law $P(t) = e^{-\Gamma t}$, is only an approximation [1]. The quantum mechanical survival probability, without any approximation is,

$$P(t) = |A(t)|^2 = |\langle \Psi | e^{-iHt} | \Psi \rangle|^2. \quad (1)$$

It can be calculated at short times to give, $P(t) \simeq 1 - (\Delta_\Psi H)t^2$, which in turn is connected to $\frac{dP(t)}{dt}|_{t=0} = 0 \leftrightarrow \frac{d(e^{-\Gamma t})}{dt}|_{t=0} \neq 0$. A direct deviation from the exponential decay law at short times has been experimentally verified [2]. It is therefore correct to say that we have a very good understanding of the first few moments in the life of an unstable quantum state. This is not the case for the large time tail as not only has it never been experimentally seen (in spite of efforts to detect it [3]), but its theoretical foundation seems to ‘enjoy’ different treatments [4]. Our idea presented in this talk is to use a semi-empirical method (experimental data in a theoretical formula) instead of a direct evidence for the large time behavior of the survival probability. We will show that this is also closely related to pinning down a more exact and reliable theoretical framework. To achieve the goals we need two different time concepts in Quantum Mechanics: (i) Time as a parameter and (ii) Time as an observable. The first is clearly the variable time t which appears in the Schrödinger equation and the survival probability; the second has

to do with quantum mechanical observables¹ such as delay-time, dwell-time or sojourn-time, Larmor-time, traversal-time etc. which were constructed to answer questions about the time spent in a region or the quantum collision time. In calculating the parametric time dependence, the 'observable time' will be necessary.

LARGE TIME BEHAVIOUR OF THE SURVIVAL AMPLITUDE

The Fock-Krylov Method

The Fock-Krylov method [5] is a suitable theoretical framework to study the large time behaviour of unstable systems. It relies on basic quantum mechanical results and therefore is, up to a point which we will discuss later, model-independent. We first observe that an unstable state $|\Psi\rangle$ cannot be an eigenstate to the energy i.e. $H|\Psi\rangle \neq E|\Psi\rangle$. Otherwise the survival amplitude $A(t)$ and the survival probability $P(t)$ would come out trivially to be, $A(t) = \langle\Psi|e^{-iHt}|\Psi\rangle = e^{-iEt}$ and $P(t) = 1$. Hence, assuming a continuum, $H|E\rangle = E|E\rangle$, $\langle E'|E\rangle = \delta(E' - E)$, we are entitled to expand

$$|\Psi\rangle = \int_{\text{Spect}(H)} dE a(E) |E\rangle \quad (2)$$

where,

$$\rho(E) \equiv \frac{\text{Prob}_\Psi(E)}{dE} = |\langle E|\Psi\rangle|^2 = |a(E)|^2, \quad (3)$$

is a probability density (and as such positive-definite) to find the states with energy E in the resonance. This distribution is also known as the spectral function. We can now calculate the survival amplitude to obtain

$$A(t) = \int_{\text{Spect}(H)} dE \rho(E) e^{-iEt} = \int_{E_{th}}^{\infty} \rho(E) e^{-iEt} \quad (4)$$

which turns out to be a Fourier transform of the spectral function. E_{th} is the sum of the masses of the decay products. The success of this general method hinges on the right choice of ρ . There is no general agreement in the literature on what this function should be (in the next section we make a claim about the correct choice of ρ), but a general parameterization looks like, $\rho(E) = (\text{Threshold}) \times (\text{Pole}) \times (\text{Form} - \text{factor})$, i.e.,

$$\rho(E) = (E - E_{th})^\gamma \times P(E) \times F(E). \quad (5)$$

Some comments about ρ are in order: (i) $P(E)$ has a simple pole at $z_R = E_R - i\Gamma_R/2$ which leads to the exponential decay law. More poles in the fourth quadrant of the complex E-plane would modify even the exponential part of the decay. (ii) $F(E)$ has no threshold and no pole. It is a smooth function which should go to zero for large

¹ Due to Pauli theorem there is no time operator as a conjugate variable to energy, but time-delay operators can exist nonetheless.

E. (iii) Large times t correspond in the Fourier transform to small E (in agreement with the time-energy uncertainty relation). Hence, the large time behaviour is due to the choice of γ which is often controversial. (iv) The transition region between exponential and non-exponential is partly due to the choice of the form-factor F . (v) The choice of the spectral function is not unique in the literature: e.g. often $\gamma = 0$ and $f(E) = 1$. The problem regarding the right choice of the spectral function will be discussed in the next section. But even without the explicit knowledge of ρ , one can extract valuable information from the Fock-Krylov method. We choose the closed path in the complex E -plane: $C_R = C_{\Im} + C_{\Re} + C_R^{1/4}$, starting from zero along the real axis (C_{\Re}) attaching to it a quarter of a circle with radius R ($C_R^{1/4}$) in the clockwise direction and completing the path by going upward the imaginary axis up to zero (C_{\Im}). Hence,

$$e^{-iE_{th}t} I \equiv e^{-iE_{th}t} \left(\int_{C_{\Re}} \dots + \int_{C_R^{1/4}} \dots + \int_{C_{\Im}} \dots \right), \quad (6)$$

where the dots indicate the integrand from equation (4) with an argument shifted by E_{th} since we start from 0. The integral we wish to calculate is along the real axis. I is calculated by using the residue theorem with the pole at z_R leading to the exponential decay law. We assume that for $R \rightarrow \infty$ the integral along the arc goes to zero. Thus,

$$\begin{aligned} A(t) &= A_E(t) + A_{LT}(t) \\ A_E(t) &= 2\pi i \tilde{P}(z_R) F(z_R) (z_R - E_{th})^\gamma e^{iE_{th}t} e^{-\Gamma_R t/2} = a_E(t) e^{-\Gamma_R t/2} \end{aligned} \quad (7)$$

with $\tilde{P}(z) = \lim_{z \rightarrow z_R} P(z)(z - z_R)$ and

$$\begin{aligned} A_{LT}(t) &= (\text{phase}) \times \int_0^\infty dx P(-ix + E_{th}) F(-ix + E_{th}) x^\gamma e^{-xt} \\ &\simeq (\text{phase}) \times \Gamma(\gamma + 1) P(E_{th}) F(E_{th}) \times \frac{1}{t^{\gamma+1}} = a_{LT} \frac{1}{t^{\gamma+1}} \end{aligned} \quad (8)$$

The result agrees with [6]. This is how nature slows down the exponential decay.

The transition region and critical times

One can approximately estimate the transition time from the exponential to the power law behaviour by setting $|a_E| e^{-2\Gamma_R t_0/2} \simeq |a_{LT}| \frac{1}{t_0^{\gamma+1}}$ or alternatively by determining the zeros of the function

$$\omega(\xi_0) \equiv \ln \frac{|a_E|}{|a_{LT}|} \left(\frac{\Gamma_R}{2} \right)^{-\gamma-1} + (\gamma+1) \ln \xi_0 - \xi_0, \quad \xi_0 \equiv \frac{\Gamma_R t}{2} \quad (9)$$

Note that, strictly speaking, there are three regions (see Fig. 1) and two critical times: the time at the transition from the exponential to the oscillatory region and from the oscillatory to the power law. The condition (9) can also have two zeros out of which normally

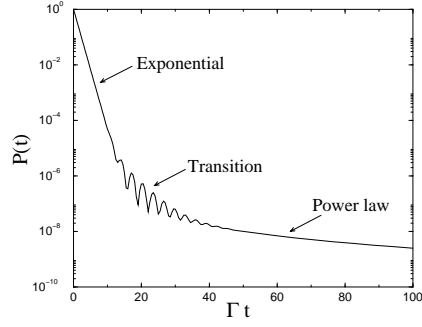


FIGURE 1. Characteristic behaviour of a survival probability.

the second one is the right indicator of critical time. Consider a narrow resonance model with the spectral function a simple Breit-Wigner, $\rho(E) \propto (E_R - E_{th})^\gamma \frac{e^{-E/E_0}}{(E - E_R)^2 + (\Gamma_R/2)^2}$. Then defining $\varepsilon \equiv E_R - E_{th}$ and choosing $\gamma = 1/2$

$$\frac{|a_E|}{|a_{LT}|(\Gamma_R/2)^{3/2}} \propto e^{-\varepsilon/E_0} [1 + \varepsilon^2/(\Gamma_R/2)^2]^{5/4}. \quad (10)$$

It is evident that smaller the ratio ε/Γ_R , smaller is the critical time². Hence for narrow resonances the best candidates to find the non-exponential long time behaviour are threshold resonances.

THE CONNECTION TO SCATTERING DATA

Beth and Uhlenbeck [7], while calculating virial coefficients B, C in the equation of an ideal gas: $pV = RT \left[1 + \frac{B}{V} + \frac{C}{V^2} + \dots \right]$, found that the difference between the density of states (of scattered particles) with interaction $dn_l(E)/dE$ and without $dn_l^{(0)}(E)/dE$ is,

$$\frac{dn}{dE} = \frac{dn_l(E)}{dE} - \frac{dn_l^{(0)}(E)}{dE} = \frac{2l+1}{\pi} \frac{d\delta_l(E)}{dE}. \quad (11)$$

In a resonant scattering [8] this is the density of states of a resonance (in terms of the decay products). For instance $T = (\Gamma_R/2)/(E_R - E - i\Gamma_R/2)$, gives,

$$\frac{d\delta}{dE} = \frac{\Gamma_R/2}{(E_R - E)^2 + \Gamma_R^2/4}. \quad (12)$$

This offers a semi-empirical method to examine the large time behaviour of unstable systems (resonances) directly from data since the spectral function is,

$$\rho(E) = \frac{d\text{Prob}}{dE} = \frac{1}{n} \frac{dn}{dE}. \quad (13)$$

² This is an approximate estimate since there are two critical times characterizing the transition region

Note that this also uniquely fixes the spectral function, at least in the vicinity of a resonance. This interpretation works well for all l - values except for the s -wave, because in this case $\frac{d\delta}{dE} \propto \frac{1}{(E-E_{th})^{1/2}}$ and we encounter a threshold singularity. This unreasonable singular behaviour of the density of states can be remedied without changing the interpretation (see next section). Otherwise for $l > 0$ we have $\frac{d\delta}{dE} \propto (E-E_{th})^{l-1/2}$ and therefore $\gamma = l - 1/2$, i.e., γ is fixed. An explicit example of $P(t)$ for narrow resonances $\alpha + \alpha \rightarrow {}^8\text{Be}(l=2) \rightarrow \alpha + \alpha$ has been calculated in [9] from scattering data given in form of the phase shift.

EXPLICIT EXAMPLES

The Fock-Krylov method is not limited to narrow resonances. We can put forth the question if there exist new features in the survival probability for broad resonances? For narrow resonances we can use a Breit-Wigner model for the amplitude to calculate the critical time and obtain some relevant results. No general parameterization of the transition amplitude for broad resonances exists. We will therefore choose the most prominent example of the σ for which some parameterizations are available. The new features which can appear here are e.g., sub-threshold zeros (called Adler's zeros) in the amplitude and hence also in the form-factor in the density of states. Besides this, the σ is an s -wave resonance. As we already mentioned the s -wave density of states has to be modified from the time delay to the dwell time delay.

$$\left(\frac{dn}{dE}\right)_{\text{new}} = \text{dwell-time} = 2\frac{d\delta}{dE} - \frac{2\Re(T)\sqrt{s}}{s-4m_\pi^2} \quad (14)$$

which is the relativistic version of the expression found in [10]. This singularity-free expression is also a density of states as shown in [11]. There exist different parameterizations of the amplitude for $\pi\pi \rightarrow \sigma \rightarrow \pi\pi$. Here we have opted for the following one from [12]:

$$T = \frac{M\Gamma(s)}{M^2 - s - iM\Gamma(s)}, \quad (15)$$

where the energy dependent width can be found in [12]. It has the structure: threshold \times Adler's zeros \times form-factor. Figure 1 displays the critical times ($\omega(\xi_0)$ from equation (9)) for very narrow resonances in nuclear physics (${}^8\text{Be}$) in contrast to the same exercise done for the broad σ . The critical lifetimes of these narrow resonances are indeed large (~ 30 and 70 in terms of lifetimes). However, in the case of a broad resonance, i.e. σ , we can conclude the following. (i) The short transition time which is usually considered an artifact of the approximation (recall that in reality there are two critical times) is the biggest we find (ca. one lifetime). The large transition time which usually is the correct time scale for transition is the smallest we find (eight lifetimes). Therefore, we think that the oscillatory transition region could here be of importance and the real transition time could be somewhere between the two we find now. This behaviour would be similar in other broad resonance systems such as the η -mesic nuclei [13]. The relatively small critical time and the importance of the oscillatory region makes the study of the time evolution of the σ an interesting undertaking which we plan to continue in future.

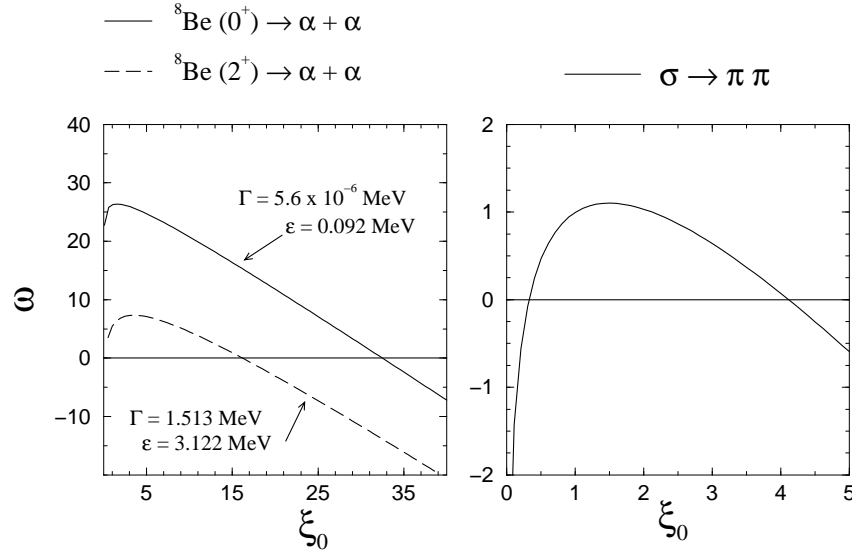


FIGURE 2. Critical times for narrow resonances and σ as an example of a broad resonance.

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